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H_3^+-WZNW correlators from Liouville theory

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Abstract: We prove that arbitrary correlation functions of the H_3^+-WZNW model on a sphere have a simple expression in terms of Liouville theory correlation functions. This is based on the correspondence between the KZ and BPZ equations, and on relations between the structure constants of Liouville theory and the H_3^+-WZNW model. In the critical level limit, these results imply a direct link between eigenvectors of the Gaudin hamiltonians and the problem of uniformization of Riemann surfaces. We also present an expression for correlation functions of the SL(2)/U(1) gauged WZNW model in terms of correlation functions in Liouville theory.

Keywords: Conformal Field Models in String Theory, Integrable Equations in Physics.

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1. Introduction

Liouville theory and the H_3^+-WZNW model (henceforth mostly abbreviated as H_3^+ model) are two conformal field theories which have played an important rôle in many recent studies of noncritical string theories, two-dimensional quantum gravity and the AdS/CFT correspondence in the case of the three-dimensional Anti-de Sitter space. These two models are examples of conformal field theories with continuous sets of primary fields which are not obtained from free field theories in any simple way. One may view Liouville theory and the H_3^+ model as "noncompact" counterparts of minimal models and the SU(2) Wess-Zumino-Novikov-Witten model respectively. These and other observations form the basis for our expectation that Liouville theory and the H_3^+ model could play a similar rôle in the development of non rational conformal field theories as the minimal and WZNW models have played in making rational conformal field theories a powerful theoretical tool with various applications. It is also worth noting that a certain number of conformal field theories with high string-theoretical relevance such as the $\mathrm{SL}(2)/\mathrm{U}(1)$ euclidean black hole model and the N=2 Liouville theory can be obtained from the (supersymmetrized) H_3^+ model by means of simple coset constructions.

The search for relations between Liouville theory and the minkowskian counterpart of the H_3^+ model, the $\mathrm{SL}(2,\mathbb{R})$ WZNW model, has a long tradition, stimulated by [1, 2]. Originally it was hoped that the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the $\mathrm{SL}(2,\mathbb{R})$ WZNW model would make it easier to solve this model first, from which quantum Liouville theory would be obtained by means of a quantum version of hamiltonian reduction [3].

Our point of view will be quite the opposite one. At the moment, it is Liouville theory which is by far the best understood example of an interacting conformal field theory with a continuous set of primary fields, see [4 − 6] and references therein. The H_3^+ model is only reasonably well understood on punctured Riemann surfaces of genus zero [7 − 10]. Not as well understood is the H_3^+ model on Riemann surfaces with boundaries. This is probably due to the fact that in contrast to the case of Liouville theory we do not understand the chiral bootstrap of the H_3^+ model properly yet.[1] Even less understood is the $\mathrm{SL}(2,\mathbb{R})$ WZNW model. Despite some important progress [11], we do not control the definition and the analytic properties of the correlation functions with more than three field insertions yet.

Our main motivation for seeking relations between Liouville theory on the one hand, and the H_3^+ model on the other hand is therefore the hope that we can use knowledge from Liouville theory to improve upon our understanding of the H_3^+ and $\mathrm{SL}(2,\mathbb{R})$ models. In the present paper we shall present an explicit formula relating arbitrary correlation functions of Liouville theory and the H_3^+ model on punctured Riemann spheres. The simplicity of this relation makes us view the H_3^+ model and its cosets like the $\mathrm{SL}(2)/\mathrm{U}(1)$ gauged WZNW model as ordinary Liouville theory in disguise. We plan to discuss similar relations between the conformal blocks of these theories in subsequent publications, and we hope that this will also allow us to make progress on the boundary problem. We also believe that similar techniques can be used to construct the correlation functions of the $\mathrm{SL}(2,\mathbb{R})$ model.

Another application that we have in mind is the investigation of the critical level limits of the H_3^+ and $\mathrm{SL}(2)/\mathrm{U}(1)$ models. It was conjectured by V. Fateev and the brothers Zamolodchikov that the $\mathrm{SL}(2)/\mathrm{U}(1)$ gauged WZNW model is dual to sine-Liouville theory. The supersymmetric counterpart of this duality can be seen as mirror symmetry [12]. These dualities should manifest themselves most clearly in the critical level limit, where the sine-Liouville and the N=2 Liouville actions would provide weakly coupled descriptions of the respective models. It is possible to construct correlation functions in all these models from the H_3^+ model [7, 13]. Given these relations it is natural to investigate if the critical level limit may also be seen as a dual weak coupling limit in the case of the H_3^+ model.

[1] It may well be that a generalization of the formalism usually called chiral bootstrap is needed in this case since the representations which generate the spectrum of the H_3^+ model do not exhibit chiral factorization in the usual sense.

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Results about the critical level limit of the H_3^+ model could then help us to deepen our understanding of the above-mentioned non-trivial dualities. Some steps in this direction will be taken in the present paper by relating the critical level limit of the H_3^+ model to the semiclassical limit of ordinary bosonic Liouville theory.

Plan of the paper. We start by collecting those results about Liouville theory and the H_3^+ model which will be relevant for us. The main novelty in this section will be to introduce a new basis for the space of primary fields in the H_3^+ model which will simplify the analysis considerably.

We then state our main result, equation (3.1). This is the expression of arbitrary n-point correlators of the H_3^+ model on the sphere in terms of Liouville correlators. We prove it by induction on n. An important ingredient is a correspondence between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov equations in a version due to Stoyanovsky, which we review. We also derive useful relations between the three-point structure constants of the models. Next we compare our main result with the original version of the KZ-BPZ correspondence due to Fateev and Zamolodchikov. We also indicate how correlators of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model are related to Liouville correlators.

The last section is devoted to the study of the critical level limit $k \downarrow 2$. In this limit, the KZ equations reduce to the eigenvector equations for the Gaudin hamiltonians. The corresponding limit in Liouville theory is known to be related to the uniformization of Riemann surfaces with conical singularities. We use known results for this problem in order to build particular eigenvectors of the Gaudin hamiltonians. The corresponding eigenvalues are given by the so-called accessory parameters of the classical Liouville solution.

2. Review of Liouville theory and the H_3^+ model

2.1 Liouville theory

Liouville theory is defined classically by the action

$$S^L = \frac{1}{\pi}\int d^2w \left(|\partial_w\phi|^2 + \mu_L e^{2b\phi}\right) . \tag{2.1}$$

Liouville theory is a conformal field theory [4, 5] whose left- and right-moving Virasoro algebras are respectively generated by the modes of the stress-energy tensors $T(w) = -(\partial\phi)^2 + Q\partial^2\phi$ and $\bar{T}(\bar{w}) = -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2\phi$. The central charge c and the background charge Q are expressed in terms of the parameter b via[2]

$$c = 1 + 6Q^2 , \qquad Q \equiv b + b^{-1} . \tag{2.2}$$

The primary fields of Liouville theory are denoted as $V_\alpha(z)$ and reduce to the exponential fields $e^{2\alpha\phi}$ in the classical limit $b \to 0$. The parameter α may take arbitrary complex values, but the subset $\alpha \in \frac{Q}{2} + i\mathbb{R}$ is distinguished by the fact that the action of $V_\alpha(z)$ on the

[2]The value of Q follows from the requirement that the Liouville interaction term $\mu_L e^{2b\phi}$ be exactly marginal.

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vacuum state $|0\rangle$ creates delta-function normalizable states. The primary field $V_\alpha(z)$ has conformal weight

$$\Delta_\alpha = \alpha(Q-\alpha)\,. \tag{2.3}$$

2.1.1 Correlation functions

The theory is fully characterized by the n-point correlators on a sphere:

$$\Omega_n^L(z_n,\dots,z_1|\alpha_n,\dots,\alpha_1) = \big\langle\, V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)\big\rangle\,. \tag{2.4}$$

Power series representations for Ω_n^L may be obtained by using the operator product expansion

$$V_{\alpha_2}(z_2)V_{\alpha_1}(z_1) = \frac{1}{2\pi}\int_{\frac{Q}{2}+i\mathbb{R}_+} d\alpha_3\, \langle V_{\alpha_3}(\infty)V_{\alpha_2}(z_2-z_1)V_{\alpha_1}(0)\rangle (V_{Q-\alpha_3}(z_1) + (\text{descendants}))\,, \tag{2.5}$$

where $V_{\alpha_3}(\infty)$ is defined so that correlation functions involving it are finite,

$$V_{\alpha_3}(\infty) \;=\; \lim_{z\to\infty} |z|^{4\Delta_\alpha}\langle 0|V_\alpha(z)\,. \tag{2.6}$$

The descendant contributions in (2.5) are fully determined by conformal covariance of the OPE, however these contributions are not known explicitly. In principle, the construction of general n-point correlation functions therefore boils down to the construction of the 2- and 3-point functions, which are of the form

$$\begin{aligned}\langle V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle \;&=\; 2\pi(\delta(Q-\alpha_2-\alpha_1) + R^L(\alpha_1)\delta(\alpha_2-\alpha_1))\,\times \\ &\quad\times|z_2-z_1|^{-4\Delta_{\alpha_1}}\,, \qquad (2.7)\\ \langle V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle \;&=\; |z_3-z_2|^{-2\Delta_{32}^1}|z_3-z_1|^{-2\Delta_{31}^2}|z_3-z_2|^{-2\Delta_{21}^3}\,\times \\ &\quad\times C^L(\alpha_3,\alpha_2,\alpha_1)\,, \qquad (2.8)\end{aligned}$$

where $\Delta_{rs}^t \equiv \Delta_{\alpha_r}+\Delta_{\alpha_s}-\Delta_{\alpha_t}$. The following expressions for the structure functions C^L and $R^L(\alpha)$ were proposed in [14, 15]:

$$\begin{aligned}C^L(\alpha_3,\alpha_2,\alpha_1) \;&=\; \frac{\Big[\pi\mu_L\gamma(b^2)b^{2-2b^2}\Big]^{b^{-1}(Q-\alpha_1-\alpha_2-\alpha_3)}}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)}\,\times \qquad (2.9)\\ &\quad\times\frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}\\ R^L(\alpha) \;&=\; (\pi\mu_L\gamma(b^2))^{\frac{Q-2\alpha}{b}}\frac{\Gamma(1+b(2\alpha-Q))}{\Gamma(1-b(2\alpha-Q))}\frac{\Gamma(1+b^{-1}(2\alpha-Q))}{\Gamma(1-b^{-1}(2\alpha-Q))}\,. \qquad (2.10)\end{aligned}$$

The quantity $R^L(\alpha)$ is the reflection coefficient of Liouville theory. It appears in the reflection relation,

$$V_\alpha = R(\alpha)V_{Q-\alpha}\,. \tag{2.11}$$

The existence of such a relation explains why the physical spectrum is generated by the states with $\alpha \in \frac{Q}{2}+i\mathbb{R}_+$ instead of $\alpha\in\frac{Q}{2}+i\mathbb{R}$, a fact we have already used in writing the OPE (2.5).

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Definition and relevant properties of the special function $\Upsilon_b(x)$ are recalled in appendix B. Consistency of the conformal field theory characterized by equations of (2.5)-(2.10) was proven in [5]. There is ample evidence [15, 4, 16] that this conformal field theory is a quantization of the classical theory described by the action (2.1) with $V_\alpha(z)$ being the quantized counterpart of $e^{2\alpha\phi(z)}$.

2.1.2 Degenerate fields and BPZ equations

The Liouville correlators Ω_n^L (2.4) are a priori defined only for physical values of the momenta, $\alpha_i \in \frac{Q}{2} + i\mathbb{R}$. However, they possess a meromorphic continuation to arbitrary complex momenta $\alpha_i \in \mathbb{C}$ [4]. In particular, we may specialize to correlators involving fields with momentum $\alpha = -\frac{1}{2b}$ which correspond to the degenerate Verma module for the Virasoro algebra with Kac labels $(1, 2)$,

$$\Omega_{n|m}^L \equiv \left\langle V_{\alpha_n}(z_n) \cdots V_{\alpha_1}(z_1)\, V_{-\frac{1}{2b}}(y_m) \cdots V_{-\frac{1}{2b}}(y_1) \right\rangle . \tag{2.12}$$

Each degenerate field $V_{-\frac{1}{2b}}(y_r)$ gives rise to a Belavin-Polyakov-Zamolodchikov (BPZ) differential equation for the correlator $\Omega_{n|m}^L$,

$$\mathrm{D}_r^{\mathrm{BPZ}}\, \Omega_{n|m}^L = 0\,, \tag{2.13}$$

where the differential operator $\mathrm{D}_r^{\mathrm{BPZ}}$ is defined as

$$\mathrm{D}_r^{\mathrm{BPZ}} = b^2 \frac{\partial^2}{\partial y_r^2} + \sum_{s \neq r} \left(\frac{1}{y_r - y_s} \frac{\partial}{\partial y_s} + \frac{\Delta_{-\frac{1}{2b}}}{(y_r - y_s)^2} \right) + \sum_s \left(\frac{1}{y_r - z_s} \frac{\partial}{\partial z_s} + \frac{\Delta_{\alpha_s}}{(y_r - z_s)^2} \right) . \tag{2.14}$$

The differential equations (2.13) express the decoupling of the singular vector in the Verma module with highest weight $\Delta_{-1/2b}$. The equations (2.13) imply in particular that the OPE $V_{-\frac{1}{2b}}(y)V_\alpha(z)$ takes the simple form

$$V_{-\frac{1}{2b}}(y)V_\alpha(z) = \sum_{\eta = \pm} |y - z|^{\eta(2\alpha - Q) + Q} C_\eta^L(\alpha) V_{\alpha - \frac{\eta}{2b}}(z) + (\text{descendants})\,, \tag{2.15}$$

where the OPE coefficients are

$$\begin{aligned} C_-^L(\alpha) &= R^L(\alpha) R^L\left(Q - \alpha - \frac{1}{2b}\right) = (\pi \mu_L \gamma(b^2))^{\frac{1}{b^2}} b^{-4} \frac{\gamma(2b^{-1}\alpha - 1 - b^{-2})}{\gamma(2b^{-1}\alpha)}\,, \\ C_+^L(\alpha) &= 1\,. \end{aligned} \tag{2.16}$$

2.2 The H_3^+ model

The classical H_3^+ model is defined by the following action:

$$S^H = k \int d^2 z \left(\partial\phi \bar{\partial}\phi + e^{2\phi} \partial\gamma \bar{\partial}\bar{\gamma} \right) . \tag{2.17}$$

The quantum H_3^+-model [8]-[10] is a conformal field theory that has the current algebra $(\widehat{\mathfrak{sl}}_2)_k \times (\widehat{\mathfrak{sl}}_2)_k$ as a symmetry. The corresponding currents will be denoted by $J^a(z)$ and $\bar{J}^a(\bar{z})$ respectively.

The primary fields of the H_3^+-model are usually parametrized as $\Phi^j(x|z)$, $z \in \mathbb{C}$, $x \in \mathbb{C}$, and are the quantized counterparts of the following functions on H_3^+:

$$\Phi^j_{\rm cl}(x|z) \equiv \frac{2j+1}{\pi}\left(|\gamma - x|^2 e^{\phi} + e^{-\phi}\right)^{2j} . \tag{2.18}$$

The conformal weight of the primary field $\Phi^j(x|z)$ is

$$\Delta_j = -b^2 j(j+1) = -\frac{j(j+1)}{k-2} , \tag{2.19}$$

where we have introduced the notation $b^2 = 1/(k-2)$. This notation anticipates the relation with the b parameter of Liouville theory.

Our review of the H_3^+ model and our analysis of its correspondence with Liouville theory will be greatly simplified by the introduction of a new set of primary fields $\Phi^j(\mu|z)$, which are defined from $\Phi^j(x|z)$ by a change of basis within each representation of spin j,

$$\Phi^j(\mu|z) = \frac{1}{\pi}|\mu|^{2j+2}\int_{\mathbb{C}} d^2x \; e^{\mu x - \bar{\mu}\bar{x}} \Phi^j(x|z) . \tag{2.20}$$

The reader may find the usual x-basis quantities, and the derivation of our new μ-basis quantities, in appendix A.

The fields $\Phi^j(\mu|z)$ are characterized by the following operator product expansions with the currents:

$$J^a(w)\Phi^j(\mu|z) \sim \frac{1}{w-z}\mathrm{D}^a\Phi^j(\mu|z) , \qquad \bar{J}^a(\bar{w})\Phi^j(x|z) \sim \frac{1}{\bar{w}-\bar{z}}\bar{\mathrm{D}}^a\Phi^j(x|z) . \tag{2.21}$$

The differential operators D^a_j are defined as

$$\mathrm{D}^- = \mu , \qquad \mathrm{D}^0 = -\mu\partial_\mu , \qquad \mathrm{D}^+ = \mu\partial_\mu^2 - \frac{j(j+1)}{\mu} , \tag{2.22}$$

and the quadratic Casimir of the corresponding representation is

$$\mathrm{D}^a\mathrm{D}^a = (\mathrm{D}^0)^2 - \frac{1}{2}(\mathrm{D}^+\mathrm{D}^- + \mathrm{D}^-\mathrm{D}^+) = j(j+1) . \tag{2.23}$$

The operators $\bar{\mathrm{D}}^a_j$ are the complex conjugates of D^a_j.

We may consider arbitrary complex values of j, but the values $j \in -\frac{1}{2} + i\mathbb{R}$ are distinguished by the fact that the action of $\Phi^j(\mu|z)$ on the vacuum $|0\rangle$ creates delta-function normalizable states, which generate the physical spectrum of the H_3^+ model. These values of j are those which appear in the decomposition of the classical limit $L^2(H_3^+)$ of the spectrum. This classical limit is defined by $k \to \infty$.

2.2.1 Correlation functions

The H_3^+-model is fully characterized by the n-point correlation functions on a sphere:

$$\Omega_n^H(\mu_n, \ldots, \mu_1|z_n, \ldots, z_1|j_n, \ldots, j_1) = \left\langle \Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_1}(\mu_1|z_1)\right\rangle . \tag{2.24}$$

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These correlation functions may be constructed by using the operator product expansion

$$\Phi^{j_2}(\mu_2|z_2)\Phi^{j_1}(\mu_1|z_1) = \int_{-\frac{1}{2}+i\mathbb{R}_+} dj_3 \int \frac{d^2\mu_3}{|\mu_3|^2} \left\langle \Phi^{j_3}(\mu_3|\infty)\Phi^{j_2}(\mu_2|z_2-z_1)\Phi^{j_1}(\mu_1|0)\right\rangle \times \\ \times \big(\Phi^{-j_3-1}(\mu_3|z_1) + (\text{descendants})\big)\,. \tag{2.25}$$

The descendant contributions are determined by the $(\widehat{\mathfrak{sl}}_2)_k \times (\widehat{\mathfrak{sl}}_2)_k$ current algebra symmetry. The construction of n-point functions is thereby in principle reduced to the construction of two- and three-point functions:

$$\langle \Phi^{j_2}(\mu_2|z_2)\Phi^{j_1}(\mu_1|z_1)\rangle = |z_2 - z_1|^{-4\Delta_{j_1}}|\mu_1|^2\delta^{(2)}(\mu_2+\mu_1)\times \\ \times\big(\delta(j_2+j_1+1) + R^H(j_1)\delta(j_2-j_1)\big) \tag{2.26}$$

$$\langle \Phi^{j_3}(\mu_3|z_3)\cdots\Phi^{j_1}(\mu_1|z_1)\rangle = |z_3-z_2|^{-2\Delta_{32}^1}|z_3-z_1|^{-2\Delta_{31}^2}|z_2-z_1|^{-2\Delta_{21}^3}\times \\ \times\delta^{(2)}(\mu_3+\mu_2+\mu_1)D^H\left[\begin{smallmatrix} j_3 & j_2 & j_1 \\ \mu_3 & \mu_2 & \mu_1\end{smallmatrix}\right] C^H(j_3,j_2,j_1)\,. \tag{2.27}$$

The structure constant $C^H(j_3,j_2,j_1)$ has the following expression (whose normalization is discussed in [9]),

$$C^H(j_3,j_2,j_1) = -\frac{1}{2\pi^3 b}\left[\frac{\gamma(b^2)b^{2-2b^2}}{\pi}\right]^{-2-\Sigma j_i} \frac{\Upsilon_b'(0)}{\Upsilon_b(-b(j_1+j_2+j_3+1))}\times \\ \times\frac{\Upsilon_b(-b(2j_1+1))\Upsilon_b(-b(2j_2+1))\Upsilon_b(-b(2j_3+1))}{\Upsilon_b(-bj_{12}^3)\Upsilon_b(-bj_{13}^2)\Upsilon_b(-bj_{23}^1)}\,, \tag{2.28}$$

where we used the notation $j_{st}^r \equiv j_s + j_t - j_r$. The quantity $\delta^{(2)}(\mu_3+\mu_2+\mu_1)D^H\left[\begin{smallmatrix} j_3 & j_2 & j_1 \\ \mu_3 & \mu_2 & \mu_1\end{smallmatrix}\right]$ is the Clebsch-Gordan coefficient for $\mathrm{SL}(2,\mathbb{C})$ representations of spins j_1, j_2, j_3 in the μ basis, which is symmetric with respect to permutations of the indices $1,2,3$ and invariant under μ-dilatations. Explicitly, we have

$$D^H\left[\begin{smallmatrix} j_3 & j_2 & j_1 \\ \mu_3 & \mu_2 & \mu_1\end{smallmatrix}\right] = \pi|\mu_2|^{-2j_1-2j_3-2}|\mu_1|^{2j_1+2}|\mu_3|^{2j_3+2}\times \tag{2.29}$$
$$\times\left[\frac{\gamma(j_{23}^1+1)\gamma(j_{13}^2+1)}{\gamma(-j_{123}-1)\gamma(2j_3+2)}\,{}_2\mathcal{F}_1\left(j_{123}+2, j_{13}^2+1, 2j_3+2, 1+\frac{\mu_1}{\mu_2}\right) + \right. \\ \left. + \left|1+\frac{\mu_1}{\mu_2}\right|^{-2(2j_3+1)} \frac{\gamma(j_{12}^3+1)}{\gamma(-2j_3)}\,{}_2\mathcal{F}_1\left(-j_{23}^1, j_{12}^3+1, -2j_3, 1+\frac{\mu_1}{\mu_2}\right)\right],$$

where we used the notations $j_{rst} = j_r + j_s + j_t$ and

$${}_2\mathcal{F}_1(a,b,c,z) = F(a,b,c,z)F(a,b,c,\bar{z})\,. \tag{2.30}$$

The reflection coefficient $R^H(j)$ is

$$R^H(j) = -\left(\frac{1}{\pi}b^2\gamma(b^2)\right)^{-(2j+1)} \frac{\Gamma(+2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(+b^2(2j+1))}{\Gamma(-b^2(2j+1))}\,. \tag{2.31}$$

This coefficient is involved in the reflection relation

$$\Phi^j(\mu|z) = R^H(j)\Phi^{-j-1}(\mu|z)\,. \tag{2.32}$$

Consistency of the conformal field theory characterized by these data was proven in [10]. There is good evidence that this conformal field theory is a quantization of the the classical theory described by the action (2.17).

2.2.2 The KZ equations

Due to the $(\widehat{\mathfrak{sl}}_2)_k \times (\widehat{\mathfrak{sl}}_2)_k$ symmetry of the H_3^+ model, the correlation functions Ω_n^H may alternatively be characterized as particular solutions to the Knizhnik-Zamolodchikov (KZ) system of partial differential equations which are subject to certain asymptotic conditions. The KZ equations for the case at hand may be written as

$$
\begin{aligned}
(k-2)\frac{\partial}{\partial z_r}\Omega_n^H &= \mathsf{H}_r\,\Omega_n^H\,, \qquad \mathsf{H}_r \equiv \sum_{s\neq r}\frac{\mathrm{D}_{rs}}{z_r - z_s}\,,\\
(k-2)\frac{\partial}{\partial \bar{z}_r}\Omega_n^H &= \overline{\mathsf{H}}_r\,\Omega_n^H\,, \qquad \overline{\mathsf{H}}_r \equiv \sum_{s\neq r}\frac{\overline{\mathrm{D}}_{rs}}{\bar{z}_r - \bar{z}_s}\,,
\end{aligned}
\tag{2.33}
$$

where the differential operator D_{rs} is defined as

$$
\begin{aligned}
\mathrm{D}_{rs} &\equiv \mathrm{D}_r^0\mathrm{D}_s^0 - \frac{1}{2}\left(\mathrm{D}_r^+\mathrm{D}_s^- + \mathrm{D}_r^-\mathrm{D}_s^+\right)\\
&= \mu_r\mu_s\left[-\left(\partial_{\mu_r} - \partial_{\mu_s}\right)^2 + \frac{j_r(j_r+1)}{\mu_r^2} + \frac{j_s(j_s+1)}{\mu_s^2}\right],
\end{aligned}
\tag{2.34}
$$

while $\overline{\mathrm{D}}_{rs}$ is the complex conjugate of D_{rs}. In addition to the equations (2.33) we shall consider the corresponding complex conjugate equations.

For later use, let us explain more precisely how the KZ equations characterize the H_3^+ correlators. The main point is that these equations are first order in ∂/z_1. Thus, the correlator Ω_n^H is characterized by its behaviour for $z_1 \to z_2$. Using the OPE, this behaviour is determined by lower correlators Ω_{n-1}^H. These remarks can be expressed more generally as the following Lemma:

Lemma. *Let $\Theta_{n-1}^H(M'|Z'|J')$ be a solution of the system of KZ equations with variables*

$$
M' = (\mu_n, \dots, \mu_3, -\mu_1 - \mu_2)\,, \qquad Z' = (z_n, \dots, z_3, z_1)\,, \qquad J' = (j_n, \dots, j_3, -j-1)\,,
$$

and let the functions $D(j_3|j_2, j_1)$ be arbitrary. Then there exists a unique solution $\Theta_n \times (M\,|\,Z\,|\,J)$,

$$
M = (\mu_n, \dots, \mu_1)\,, \qquad Z = (z_n, \dots, z_1)\,, \qquad J = (j_n, \dots, j_1)\,,
$$

to the KZ-equations (2.33) with the asymptotic behavior

$$
\begin{aligned}
\Theta_n(X|Z|J) = \int_{-\frac{1}{2}+i\mathbb{R}_+} & dj_{21}\, D(j_{21}|j_2, j_1)\, |z_2 - z_1|^{2(\Delta_{j_{21}} - \Delta_{j_2} - \Delta_{j_1})} \times\\
&\times D^H\left[\begin{smallmatrix} j_{21} & j_2 & j_1 \\ -\mu_1-\mu_2 & \mu_2 & \mu_1 \end{smallmatrix}\right]\left(\Theta_{n-1}^H(M'|Z'|J') + \mathcal{O}(z_2 - z_1)\right).
\end{aligned}
\tag{2.35}
$$

3. H_3^+ correlators from Liouville theory

The main result of this paper is a relation between a generic H_3^+ correlator on the sphere and Liouville correlators on the sphere involving degenerate fields:

$$
\boxed{
\begin{aligned}
\left\langle \Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_1}(\mu_1|z_1)\right\rangle =& \frac{\pi}{2}(-\pi)^{-n} b\,\delta^{(2)}\left(\sum_{i=1}^n \mu_i\right) |\Theta_n|^2 \times\\
&\times \left\langle V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1) V_{-\frac{1}{2b}}(y_{n-2})\cdots V_{-\frac{1}{2b}}(y_1)\right\rangle.
\end{aligned}
}
$$

The function Θ_n which appears here is defined as

$$\Theta_n(z_1,\ldots,z_n|y_1,\ldots,y_{n-2},u) = u \prod_{r<s\leq n} z_{rs}^{\frac{1}{2b^2}} \prod_{k<l\leq n-2} y_{kl}^{\frac{1}{2b^2}} \prod_{r=1}^{n}\prod_{k=1}^{n-2}(z_r-y_k)^{-\frac{1}{2b^2}}\,. \tag{3.1}$$

The relation (3.1) will hold provided that the respective variables are related as follows:

1. The variables $\mu_1,\ldots,\mu_n$ are related to $y_1,\ldots,y_{n-2},u$ via

$$\boxed{\sum_{i=1}^{n}\frac{\mu_i}{t-z_i} = u\frac{\prod_{j=1}^{n-2}(t-y_j)}{\prod_{i=1}^{n}(t-z_i)}}\,. \tag{3.2}$$

   In particular, since $\sum_{i=1}^n \mu_i = 0$, we have $u = \sum_{i=1}^n \mu_i z_i$.

2. The Liouville parameter b is identified with the H_3^+ parameter $b^2 = (k-2)^{-1}$.

3. The Liouville bulk coupling is fixed to the value $\mu_L = b^2/\pi^2$.

4. The Liouville momenta are given by

$$\alpha_i = b(j_i+1) + \frac{1}{2b}\,. \tag{3.3}$$

The key element in the correspondence (3.1) is clearly the change of variables (3.2). The relation (3.2) can in principle be solved to express the variables y_r, $r=1,\ldots,n-2$ in terms of the μ_s, $s=1,\ldots,n$. The solution will be unique up to permutations of the variables y_r. This ambiguity will not jeopardize the validity of (3.1) since the right hand side is symmetric under permutations of the y_r, $r=1,\ldots,n-2$ thanks to the mutual locality of the fields $V_{-\frac{1}{2b}}(y_r)$.

3.1 Proof of the main result

The proof will be carried out in three steps. The induction on the number of H_3^+ fields n will first be initiated by a check of our main result in the cases of the two- and three-point H_3^+ correlators. The second step will be to prove the equivalence of the KZ and BPZ equations satisfied respectively by the H_3^+ and Liouville correlators which appear in our main result eq. (3.1). This will then be used in the third step, the inductive proof of our main result for arbitrary n.

3.1.1 Step 1: direct proof for the cases $n = 2, 3$

The case $n = 2$ is completely straightforward, since it reduces to comparing the Liouville (2.7) and H_3^+ (2.26) two-point functions. These turn out to be equivalent due to the relation between the reflection amplitudes of the models,

$$R^L(b(j+1)+\frac{1}{2b}) = R^H(j)\,. \tag{3.4}$$

Note also the relation between conformal dimensions, which is valid if $\alpha = b(j+1) + \frac{1}{2b}$:

$$\Delta_j = \Delta_\alpha - \frac{1}{2} - \frac{1}{4b^2} = \Delta_\alpha + \Delta_{-\frac{1}{2b}} + \frac{1}{2b^2}\,. \tag{3.5}$$

In the case $n = 3$ we start with the Liouville side of the main result (3.1), which involves a degenerate Liouville four-point correlator

$$\left\langle V_{\alpha_3}(z_3) V_{-\frac{1}{2b}}(y_1) V_{\alpha_2}(z_2) V_{\alpha_1}(z_1) \right\rangle .$$

This can be computed by using the operator product expansion involving a degenerate field (2.15). The result can be written as

$$\begin{aligned}
&\left\langle V_{\alpha_3}(z_3) V_{-\frac{1}{2b}}(y_1) V_{\alpha_2}(z_2) V_{\alpha_1}(z_1) \right\rangle = \\
&\quad = |z_3 - z_2|^{2(\Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3} + \Delta_{-\frac{1}{2b}})} |z_3 - z_1|^{2(\Delta_{\alpha_2} - \Delta_{\alpha_1} - \Delta_{\alpha_3} + \Delta_{-\frac{1}{2b}})} \times \\
&\quad\quad \times |z_2 - z_1|^{2(\Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{-\frac{1}{2b}})} |z_3 - y_1|^{-4\Delta_{-\frac{1}{2b}}} |1 - z|^{2(j_1+1)+b^{-2}} \times \\
&\quad\quad \times \sum_{\eta=\pm} C^L_\eta(\alpha_3) C^L\left(\alpha_2, \alpha_1, \alpha_3 - \frac{\eta}{2b}\right) |z|^{2(\Delta_{\alpha_3 - \frac{\eta}{2b}} - \Delta_{\alpha_3} + \Delta_{-\frac{1}{2b}})} \times \\
&\quad\quad \times {}_2\mathcal{F}_1(-j_3^\eta + j_1 + j_2 + 1, -j_3^\eta + j_1 - j_2, -2j_3^\eta, z)\,.
\end{aligned}$$

We have (partially) expressed the variables α_i in terms of the j_i by using (3.3), and otherwise used the notations

$$z \equiv \frac{(z_1 - z_2)(y_1 - z_3)}{(z_1 - z_3)(y_1 - z_2)}\,, \qquad j^\eta \equiv \begin{cases} j & \eta = -\,, \\ -j-1 & \eta = +\,. \end{cases}$$

We still need to rewrite the Liouville structure constants appearing in eq. (3.6) in terms of H_3^+ structure constants. This is done by using the identities

$$\begin{aligned}
C^L\left(\alpha_2, \alpha_1, \alpha_3 + \frac{1}{2b}\right) C^L_-(\alpha_3) &= -\frac{2\pi^3}{b} \frac{\gamma(j_{12}^3 + 1)}{\gamma(-2j_3)} C^H(j_3, j_2, j_1)\,, && (3.6)\\
C^L\left(\alpha_2, \alpha_1, \alpha_3 - \frac{1}{2b}\right) C^L_+(\alpha_3) &= -\frac{2\pi^3}{b} \frac{\gamma(j_{32}^1 + 1)\gamma(j_{31}^2 + 1)}{\gamma(-j_{123} - 1)\gamma(2j_3 + 2)} C^H(j_3, j_2, j_1)\,. && (3.7)
\end{aligned}$$

Furthermore, we can perform the change of variables (3.2), which leads in particular to the following expression for y_1 and the cross-ratio z:

$$y_1 = -\frac{\mu_1 z_2 z_3 + \mu_2 z_3 z_1 + \mu_3 z_1 z_2}{\sum_{i=1}^n \mu_i z_i}\,, \qquad z = 1 + \frac{\mu_1}{\mu_2}\,. \tag{3.8}$$

The two terms $\eta = \pm$ of eq. (3.6) thus combine into the two terms of the H_3^+ structure constant D^H, equation (2.29)

$$\begin{aligned}
&\left\langle V_{\alpha_3}(z_3) V_{-\frac{1}{2b}}(y_1) V_{\alpha_2}(z_2) V_{\alpha_1}(z_1) \right\rangle = \\
&\quad = |z_{32}|^{2(\Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3} - \Delta_{-\frac{1}{2b}})} |z_{31}|^{2(\Delta_{\alpha_2} - \Delta_{\alpha_1} - \Delta_{\alpha_3} - \Delta_{-\frac{1}{2b}})} |z_{21}|^{2(\Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{-\frac{1}{2b}})} \times \\
&\quad\quad \times |u|^{4\Delta_{-\frac{1}{2b}}} |\mu_1|^{\frac{1}{b^2}} |\mu_2|^{\frac{1}{b^2}} |\mu_3|^{\frac{1}{b^2}} - \frac{2\pi^2}{b} C^H(j_3, j_2, j_1)\, D^H\left[\begin{smallmatrix} j_3 & j_2 & j_1 \\ \mu_3 & \mu_2 & \mu_1 \end{smallmatrix}\right].
\end{aligned} \tag{3.9}$$

This has to be multiplied by

$$|\Theta_3(z_1,z_2,z_3|y,u)|^2 = |u|^{-4\Delta-\frac{1}{2b}}|\mu_1|^{-\frac{1}{b^2}}|\mu_2|^{-\frac{1}{b^2}}|\mu_3|^{-\frac{1}{b^2}}|z_{32}|^{-\frac{1}{b^2}}|z_{31}|^{-\frac{1}{b^2}}|z_{21}|^{-\frac{1}{b^2}}\,. \tag{3.10}$$

Taking into account the relation for conformal dimensions (3.5), the result is the desired H_3^+ three-point function appearing in the main result (3.1).

3.1.2 Step 2: correspondence between the differential equations

We want to prove that validity of the BPZ equations for the Liouville correlation function appearing in (3.1) is equivalent to the fact that the right hand side of (3.1) satisfies the KZ equations. This observation is essentially due to Stoyanovsky [27]. For the reader's convenience we shall explain the proof of this claim.[3] To begin with, let us rewrite the system of KZ equations (2.33) in an equivalent form by introducing an arbitrary parameter t and taking linear combinations of the KZ equations:

$$\sum_{r=1}^{n}\frac{k-2}{t-z_r}\left(\frac{\partial}{\partial z_r}-\frac{\Delta_{j_r}}{t-z_r}\right)\Omega_n^H \;=\; \mathsf{S}(t)\,\Omega_n^H\,, \tag{3.11}$$

where $\mathsf{S}(t)$ is defined as

$$\mathsf{S}(t)=\sum_{r=1}^{n}\left(\frac{\mathsf{H}_r}{t-z_r}+\frac{j_r(j_r+1)}{(t-z_r)^2}\right)=(\mathsf{J}^0(t))^2+\partial_t\mathsf{J}^0(t)-\mathsf{J}^-(t)\mathsf{J}^+(t)\,. \tag{3.12}$$

We have written the result in terms of $\mathsf{J}^a(t)=\sum_{r=1}^n\frac{\mathrm{D}_r^a}{t-z_r}$ because we are now interested in the case $t=y_a$ where we have $\mathsf{J}^-(y_a)=0$ due to the relation (3.2). This relation also implies

$$\frac{\partial}{\partial y_a}\;=\;\sum_{r=1}^{n}\frac{\mu_r}{y_a-z_r}\frac{\partial}{\partial\mu_r}=-\mathsf{J}^0(y_a)\,,$$

and thus we have

$$\mathsf{S}(t)\big|_{t=y_a}=\left(\frac{\partial}{\partial y_a}\right)^2. \tag{3.13}$$

This identity is a nice observation originally due to Sklyanin. It implies that the eigenvalue problem for the Gaudin hamiltonians $\mathsf{H}_r\Psi=E_r\Psi$ is equivalent to the system of $n-2$ separated equations

$$\partial_{y_a}^2\Psi=\sum_{r=1}^{n}\left(\frac{E_r}{y_a-z_r}+\frac{j_r(j_r+1)}{(y_a-z_r)^2}\right)\Psi\,. \tag{3.14}$$

We note that the equation with index a does not contain any reference to the variables y_b, $b\neq a$. The transition from the original multidimensional eigenvalue problem to a set of decoupled one-dimensional problems is called the separation of variables.

[3]There are two differences with Stoyanovsky here. First, we impose $\sum_{i=1}^n\mu_i=0$. This reduces the number of Liouville degenerate fields to $n-2$ instead of $n-1$. Second, our normalization of the operators $\Phi^j(\mu)$ eq. (2.20) leads to notable simplifications, in particular our function Θ_n does not depend on the spins j_r.

It remains to consider the left hand side of the KZ equations in the form (3.11), specialized to the values $t = y_a$. We again use the change of variables (3.2) in the form

$$\mu_r = u\frac{\prod_{a=1}^{n-2}(z_r - y_a)}{\prod_{s\neq r}(z_r - z_s)}\,. \tag{3.15}$$

This allows us to derive the identity

$$\delta_a\mu_s = 0\,, \qquad \delta_a \equiv \sum_{r=1}^{n}\frac{1}{y_a - z_r}\left(\frac{\partial}{\partial z_r} + \frac{\partial}{\partial y_a}\right) - \sum_{b\neq a}\frac{1}{y_a - y_b}\left(\frac{\partial}{\partial y_a} - \frac{\partial}{\partial y_b}\right). \tag{3.16}$$

With the help of relations (3.13) and (3.16) it becomes easy to see that the KZ equations are equivalent to the system of equations

$$(k-2)\left(\delta_a - \sum_{r=1}^{n}\frac{\Delta_{j_r}}{(y_a - z_r)^2}\right)\Omega_n^H \;=\; \partial_{y_a}^2\,\Omega_n^H\,. \tag{3.17}$$

It is then straightforward to check that twisting by the function Θ_n (3.1) yields the BPZ equations, plus the worldsheet translation invariance equation

$$\sum_{r=1}^{n}\frac{\partial}{\partial z_r} + \sum_{a=1}^{n-2}\frac{\partial}{\partial y_a} = 0\,. \tag{3.18}$$

3.1.3 Step 3: generalization to arbitrary n

Let us assume that our main result (3.1) has been proven for all $n' < n$ (with $n \geq 4$). We will show that this implies the validity of (3.1) for $n' = n$. Since we now know that both sides in (3.1) satisfy the same first-order differential equations in z_r, it is enough to show that they are equal in the limit $z_{12} \to 0$ (see the Lemma 1). In this limit, the OPE (2.25) reduces the H_3^+ correlator Ω_n^H to $(n-1)$-point and 3-point correlators:

$$\begin{aligned}&\left\langle \Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_1}(\mu_1|z_1)\right\rangle = \\ &\quad= \int_{-\frac{1}{2}+i\mathbb{R}_+} dj_{21}\int\frac{d^2\mu_{21}}{|\mu_{21}|^2}\left\langle\Phi^{j_{21}}(\mu_{21}|\infty)\Phi^{j_2}(\mu_2|z_2)\Phi^{j_1}(\mu_1|z_1)\right\rangle\times \\ &\quad\times\left(\left\langle\Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_3}(\mu_3|z_3)\Phi^{-j_{21}-1}(\mu_{21}|z_1)\right\rangle + \mathcal{O}(z_{21})\right).\end{aligned} \tag{3.19}$$

We need to compare this with the limit $z_{12} \to 0$ of the right hand side of equation (3.1), which is

$$\frac{\pi}{2}(-\pi)^{-n}b\;\delta^{(2)}\left(\sum_{i=1}^{n}\mu_i\right)|\Theta_n|^2\left\langle V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_{n-2})\cdots V_{-\frac{1}{2b}}(y_1)\right\rangle. \tag{3.20}$$

In order to determine the behaviour of this quantity, we have to study the behaviour of the y_as. Relation (3.2) implies that in the limit $z_2 - z_1 \to 0$ one of the variables y_a, henceforth taken to be y_1, will also approach z_1,

$$y_1 = z_1 + (z_2 - z_1)\frac{\mu_1}{\mu_1 + \mu_2} + \mathcal{O}((z_2 - z_1)^2)\,. \tag{3.21}$$

The remaining y_as are defined through

$$\frac{\mu_1+\mu_2}{t-z_1}+\sum_{r=3}^{n}\frac{\mu_r}{t-z_r}=-\frac{\prod_{a=1}^{n-2}(t-y_a)}{(t-z_1)\prod_{r=3}^{n}(t-z_r)}\sum_{r=1}^{n}z_r\mu_r\,. \tag{3.22}$$

We therefore need to study the asymptotic behavior of the Liouville correlator

$$\left\langle \cdots V_{-\frac{1}{2b}}(y_1)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\right\rangle$$

in the limit $z_{12},\ y_1-z_1\to 0$. A generalization of the operator product expansion (2.5) leads to:

$$\begin{aligned}
&\left\langle V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_m)\cdots V_{-\frac{1}{2b}}(y_1)\right\rangle = \\
&=\frac{1}{2\pi}\int_{\frac{Q}{2}+i\mathbb{R}_+}d\alpha_{21}\langle V_{Q-\alpha_{21}}(\infty)\,V_{-\frac{1}{2b}}(y_1)V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)\rangle\times\\
&\quad\times\left(\left\langle V_{\alpha_n}(z_n)\cdots V_{\alpha_3}(z_3)V_{\alpha_{21}}(z_1)V_{-\frac{1}{2b}}(y_m)\cdots V_{-\frac{1}{2b}}(y_2)\right\rangle+\mathcal{O}(z_{21})\right),
\end{aligned}\tag{3.23}$$

where the operator $V_\alpha(\infty)$ was defined in eq. (2.6).

We now study the Θ_n factors. The following rewriting,

$$\Theta_n(z_1,\ldots,z_n|\mu_1,\ldots,\mu_n)=\left(\sum_{i=1}^{n}\mu_i z_i\right)^{1+\frac{n}{2b^2}}\prod_{r=1}^{n}\mu_r^{-\frac{1}{2b^2}}\prod_{r<s\le n}z_{rs}^{-\frac{1}{2b^2}}\prod_{k<l<\le n-2}y_{kl}^{\frac{1}{2b^2}}\,, \tag{3.24}$$

makes it easy to derive the asymptotic behaviour,

$$\begin{aligned}
\Theta_n(z_1,z_2,z_3,\ldots,z_n|\mu_1,\mu_2,\mu_3,\ldots,\mu_n)\;=\;&|\mu_1{+}\mu_2|^{-2}\times\\
&\times\Theta_{n-1}(z_1,z_3\ldots,z_n|\mu_1{+}\mu_2,\mu_3,\ldots,\mu_n)\times\\
&\times\Theta_3\big(z_1,z_2,\infty|\mu_1,\mu_2,-\mu_1-\mu_2\big)\big(1+\mathcal{O}(z_{21})\big)\,,
\end{aligned}\tag{3.25}$$

where $\Theta_3(\ldots,z_3=\infty|\ldots)$ is defined as follows:

$$\Theta_3(\ldots,z_3=\infty|\ldots)=\lim_{z_3\to\infty}z_3^{-1-\frac{1}{2b^2}}\Theta_3(\ldots,z_3|\ldots)\,.$$

The $z_{12}\to 0$ behaviour of the Liouville side eq. (3.20) is now fully determined in terms of the behaviours of Θ_n eq. (3.25) and $\Omega^L_{n|n-2}$ eq. (3.23). In order to compare it with the behaviour of the H_3^+ side eq. (3.19), we only need to apply the induction hypothesis eq. (3.1) at levels $n'=3$ and $n'=n-1$, while performing the change of integration variable $d\alpha_{21}=b\;dj_{21}$. This shows the validity of (3.1) at level n.

3.2 Comparison with the Fateev-Zamolodchikov correspondence

The first instance of a relation between KZ and BPZ equations was found by Fateev and Zamolodchikov in [25]. They found a correspondence between the KZ equation satisfied by a generic four-point H_3^+ correlator,

$$\Omega_4^H=\langle\Phi^{j_4}(\infty|\infty)\Phi^{j_3}(1|1)\Phi^{j_2}(x|z)\Phi^{j_1}(0|0)\rangle\,, \tag{3.26}$$

and the BPZ equation satisfied by a Liouville five-point correlator with one degenerate field[4]

$$\Omega^L_{4|1} = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-\frac{1}{2b}}(x) V_{\alpha_2}(z) V_{\alpha_1}(0)\rangle\,. \tag{3.28}$$

The relation holds provided the Liouville momenta are given in terms of the H_3^+ spins by

$$\begin{cases} \alpha_2 = -b(j_1+j_2+j_3+j_4+1)\,, \\ 2\alpha_i - Q = b(-j_2+j_1+j_3+j_4-2j_i) \qquad (i=1,3,4)\,. \end{cases} \tag{3.29}$$

This relation was originally found in the context of the SU(2)-WZNW model, which would correspond to negative integer values of k and half-integer spins j_r in our notations.

The Fateev-Zamolodchikov KZ-BPZ relation has had several applications. For instance, it has been extended by one of us to a relation between the corresponding physical H_3^+ and Liouville correlators in order to prove the crossing symmetry of the H_3^+ model from the crossing symmetry of Liouville theory [10]. However, the relation studied in [10] involves complicated j-dependent factors. It seems impossible to generalize the correspondences based on the Fateev-Zamolodchikov KZ-BPZ relation to arbitrary n-point correlators.

3.3 H_3^+ correlators in the m-basis and the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model

There exists a basis of H_3^+ primary fields which is particularly useful in connection with the $\mathrm{SL}(2,\mathbb{R})$ coset model:

$$\Phi^j_{m\bar m} = \int d^2x\; x^{-j-1+m}\bar x^{-j-1+\bar m}\Phi^j(x)\,. \tag{3.30}$$

where m and $\bar m$ are restricted to the set of values

$$m = \frac{n+ip}{2}\,, \quad \bar m = \frac{-n+ip}{2}\,, \qquad n\in\mathbb{Z}\,, \quad p\in\mathbb{R}\,. \tag{3.31}$$

The fields $\Phi^j_{m\bar m}$ are then related to the $\Phi^j(\mu)$ as follows

$$\Phi^j_{m\bar m} = N^j_{m\bar m}\int \frac{d^2\mu}{|\mu|^2}\mu^{-m}\bar\mu^{-\bar m}\;\Phi^j(\mu)\,, \tag{3.32}$$

where the normalization factor $N^j_{m\bar m}$ is defined as

$$N^j_{m\bar m} \equiv \frac{\Gamma(-j+m)}{\Gamma(j+1-\bar m)}\,. \tag{3.33}$$

As a check, we can derive the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model's reflection amplitude from the reflection coefficient $R^H(j)$ using this formula.

[4]The Liouville five-point correlator has to be multiplied by a factor $|\Theta_{\rm FZ}|^2$ with

$$\Theta_{\rm FZ} = x^{-\alpha_1/b}(1-x)^{-\alpha_3/b}(x-z)^{-\alpha_2/b}z^{-2(b^2j_1j_2-\alpha_1\alpha_2)}(1-z)^{-2(b^2j_3j_2-\alpha_3\alpha_2)}\,. \tag{3.27}$$

We may use (3.32) in order to translate our main result (3.1) into a representation for correlation functions of the fields $\Phi^j_{m\bar{m}}$. The integral over the variables μ_r can be transformed into an integral over $u, y_1, \dots, y_{n-2}$ by using

$$\frac{d^2\mu_1}{|\mu_1|^2}\cdots\frac{d^2\mu_n}{|\mu_n|^2}\delta^{(2)}\left(\sum_{r=1}^n \mu_r\right) = \frac{d^2u}{|u|^2}d^2y_1\dots d^2y_{n-2}\frac{\prod_{r\neq s}|z_r-z_s|^2\prod_{a\neq b}|y_a-y_b|^2}{\prod_{r=1}^n\prod_{a=1}^{n-2}|z_r-y_a|^2}. \quad (3.34)$$

This leads to the following expression for correlation functions of the fields $\Phi^j_{m\bar{m}}$ in terms of Liouville correlators:

$$\begin{aligned}
&\left\langle \Phi^{j_1}_{m_1\bar{m}_1}(z_1)\cdots\Phi^{j_n}_{m_n\bar{m}_n}(z_n)\right\rangle = \\
&\quad= \frac{2\pi^3 b}{\pi^{2n}}\,\delta\left(i\sum_{r=1}^n(m_r+\bar{m}_r)\right)\delta_{\sum_{r=1}^n(m-\bar{m})}\times \\
&\quad\quad\times\prod_{r<s\leq n}(z_r-z_s)^{m_r+m_s+\frac{k}{2}}(\bar{z}_r-\bar{z}_s)^{\bar{m}_r+\bar{m}_s+\frac{k}{2}}\prod_{r=1}^n N^{j_r}_{m_r\bar{m}_r}\times \\
&\quad\quad\times\int_{\mathbb{C}} d^2y_1\cdots d^2y_{n-2}\prod_{a<b\leq n-2}|y_a-y_b|^k\prod_{r=1}^n\prod_{a=1}^{n-2}(z_r-y_a)^{-m_r-\frac{k}{2}}(\bar{z}_r-\bar{y}_a)^{-\bar{m}_r-\frac{k}{2}}\times \\
&\quad\quad\times\left\langle V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_{n-2})\cdots V_{-\frac{1}{2b}}(y_1)\right\rangle
\end{aligned} \quad (3.35)$$

The integrals in this formula are absolutely convergent if we assume that m and $\bar{m}$ take the values (3.31).

The H_3^+ correlators in the $\Phi^j_{m\bar{m}}$ basis have a simple relation to the winding-number preserving correlators of fields creating states in the continuous spectrum of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model. It suffices to multiply our formula (3.35) with appropriate free boson correlators. For more details on this procedure, see [28] and references therein.

Now the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model has been conjectured to be dual to Sine-Liouville theory by Fateev, Zamolodchikov and Zamolodchikov. At the perturbative level, this duality is an identity between the correlation functions of the two models, once the appropriate identification of variables is performed. Therefore, our result (3.35) implies that winding-preserving Sine-Liouville correlators can be expressed in terms of Liouville correlators. It was brought to our attention that such a relation was already found a long time ago by V. Fateev [29], using free field calculations. His relation agrees with our result up to normalizations.[5]

4. Critical level limit

As an application of our previous results, we shall now consider the critical level limit $k \downarrow 2$. The corresponding limit $b \to \infty$ of the Liouville correlation functions is equivalent to the semi-classical limit $b \to 0$, as follows from the self-duality of Liouville theory under $b \to b^{-1}$. By combining known results about the semi-classical limit of Liouville theory with

[5]Moreover, Fateev also found an expression for winding-violating correlators in terms of Liouville correlators.

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our correspondence (3.1) between Liouville theory and the H_3^+ model it becomes possible to characterize the critical level asymptotics of the correlation functions in the H_3^+ model rather precisely.

In the following two subsections we will analyze the critical level limit directly within the H_3^+ model. In this limit, H_3^+ correlators can be expressed as linear combinations of Gaudin eigenvectors, but we will not be able to determine which combinations appear. This will become possible by making use of our H_3^+-Liouville correspondence in subsection 4.4. Before this, we will review the relevant properties of Liouville correlators in the semi-classical limit (subsection 4.3).

4.1 Critical level asymptotics of KZ solutions

Useful information about the structure of correlation functions of the H_3^+ model in the limit $k \downarrow 2$ can be obtained from a quasiclassical analysis of the KZ equations. In the case of solutions to the KZ equations which take values in tensor products of finite-dimensional representations of $\mathfrak{sl}_2$ such an analysis was carried out in [17]. Here, however, we will be interested in solutions which take values in infinite dimensional representations of the zero-mode subalgebra $\mathfrak{sl}_2$. More specifically, for later convenience we will be interested in representations from the *complementary* series of $\mathrm{SL}(2,\mathbb{C})$, which corresponds to real values of the spin, $-1 < j < 0$. We will furthermore be interested in solutions to both the KZ equations and their complex conjugates.

Following [17], we will look for a solution in the sense of formal power series in $k-2$ with leading terms of the form

$$e^{-\frac{1}{k-2}S(z_1\dots z_n)}\Psi(\mu_1\dots\mu_n|z_1\dots z_n)\big(1+\mathcal{O}(k-2)\big)\,. \tag{4.1}$$

Inserting (4.1) into the KZ equations (2.33), one may easily verify that these equations are solved to leading order in $k-2$ provided that

$$\text{(i)}\quad \mathsf{H}_r\Psi = E_r\Psi\,, \qquad \bar{\mathsf{H}}_r\Psi = \bar{E}_r\Psi\,, \tag{4.2}$$

$$\text{(ii)}\quad \partial_{z_r}S = -E_r\,, \qquad \partial_{\bar{z}_r}S = -\bar{E}_r\,. \tag{4.3}$$

The first of these two equations is the system of common eigenvector equations for the set of $2n$ commuting Gaudin hamiltonians $\mathsf{H}_r, \bar{\mathsf{H}}_r$ [18–20]. As expressed by equation (4.3), the $2n$ Gaudin eigenvalues all derive from a potential $S = S(z_1\dots z_n)$.

One should note, however, that in our case it does not seem to be completely straight-forward to prove that the relevant solutions neccessarily have the form proposed in (4.1). The proof of the integrability conditions for (4.3), $\partial_{z_r}E_s = \partial_{z_s}E_r$, given in [17] requires the finite-dimensionality of the relevant representations of $\mathfrak{sl}_2$. Instead of further investigating the possibility to give a direct proof of (4.1) we shall justify this ansatz a posteriori with the help of the H_3^+-Liouville correspondence. For the moment let us simply adopt (4.1) as a working hypothesis and try to see how far we get.

Like the Gaudin hamiltonians, the eigenvalues $E_r = E_r(z_1\dots z_n)$ and $\bar{E}_r$ depend on the worldsheet coordinates. The eigenvalues are not all independent, but rather restricted by the equations

$$\sum_{r=1}^{n} z_r^k \big(z_r E_r + (k+1) j_r(j_r+1)\big) = 0 \qquad \text{for } k=-1,0,1\,, \tag{4.4}$$

which express the invariance under worldsheet Möbius transformations $z_r \to \frac{az_r+b}{cz_r+d}$.

As already observed in the previous section, it is possible to perform the separation of variables in the Gaudin eigenvalue equations (4.2) by changing variables from $\mu_1,\ldots,\mu_n$ to y_a, $a=1,\ldots,n-2$, and applying Sklyanin's observation (3.13). The resulting equations

$$\frac{\partial^2}{\partial {y_a}^2}\Psi = \sum_{r=1}^{n}\left(\frac{E_r}{y_a - z_r} + \frac{j_r(j_r+1)}{(y_a-z_r)^2}\right)\Psi\,, \tag{4.5}$$

together with their complex conjugate counterparts can be solved in a factorized form,

$$\Psi = \prod_{a=1}^{n-2}\psi_a(y_a|z_1\ldots z_n)\prod_{b=1}^{n-2}\bar{\psi}_b(\bar{y}_b|z_1\ldots z_n)\,. \tag{4.6}$$

The most general solution to the separated equations (4.5) is a linear combination of factorized solutions of the form (4.6). Since the separated equations are second order, there are $2^{2(n-2)}$ independent solutions.

Notations. In this section, we omit the dependence on antiholomorphic variables $\bar{z}, \bar{\mu}$. However, for the y_a variables which arise from Sklyanin's separation of variables, no implicit dependence on $\bar{y}_a$ should be assumed unless explicitly stated.

4.2 Critical level asymptotics of H_3^+ correlators

Let us now consider the $k \downarrow 2$ asymptotic behaviour of the H_3^+ model n-point correlator Ω_n^H. A priori, this behaviour is given by a linear combination of terms of the form (4.1) for different functions $S^\ell(z_1\ldots z_n)$ (and thus different sets of Gaudin eigenvalues $E_r^\ell, \bar{E}_r^\ell$). In turn, each Gaudin eigenvector Ψ^ℓ is a linear combination of factorized solutions (4.6).

Nevertheless, it is natural to expect that locally in $(z_1\ldots z_n)$ one function $S^{\min}$ will dominate the asymptotic behaviour of Ω_n^H. This function will naturally be the one with smallest real part $\Re S^\ell$. As a working hypothesis, we also assume that this $S^{\min}$ is unique. This means

$$\Omega_n^H \underset{k\downarrow 2}{\asymp} e^{-\frac{1}{k-2}S^{\min}(z_1\ldots z_n)}\Psi(\mu_1\ldots\mu_n|z_1\ldots z_n)\,, \tag{4.7}$$

where Ψ is an eigenvector of the Gaudin hamiltonians with eigenvalues $E_r^{\min} = \partial_{z_r}S^{\min}$.

Sklyanin's observation allows us to write Ψ as a linear combination of solutions to the Fuchsian differential equations (4.5) which have the factorized form (4.6). Now let us use the single-valuedness of Ω_n^H w.r.t. the y_a variables in order to restrict the form of this linear combination. We will argue that this requirement not only restrict the coefficients which can be used to write Ψ as a linear combination of eigenvectors (4.6), they also turn out to impose severe restrictions on the eigenvalues $E_1^{\min},\ldots,E_n^{\min}$ themselves.

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Let us focus on the dependence of Ψ w.r.t. some $y \in \{y_1 \dots y_{n-2}\}$. The y-dependence of a general solution is given by

$$\chi(y,\bar{y}|z_1 \dots z_n) = \big(\bar{\psi}_1(\bar{y}), \bar{\psi}_2(\bar{y})\big) \cdot K \cdot \begin{pmatrix} \psi_1(y) \\ \psi_2(y) \end{pmatrix}, \tag{4.8}$$

where $\psi_1(y), \psi_2(y)$ are two linearly independent solutions of the Fuchsian differential equation (4.5), and K is a 2×2 matrix. Single-valuedness of $\chi(y,\bar{y})$ constrains the 2×2 monodromy matrices M_r of $\big(\bar{\psi}_1(\bar{y}), \bar{\psi}_2(\bar{y})\big)$ around z_r,

$$M_r^\dagger \cdot K \cdot M_r = K \qquad \text{for all } r = 1, \dots, n\,. \tag{4.9}$$

This is a highly overdetermined system of equations for the matrix K, which will also restrict the monodromy matrices M_r. If, for example, all M_r happen to be contained in the subgroup $\mathrm{SU}(1,1)$ of $\mathrm{SL}(2,\mathbb{C})$, we may use $K = \mathrm{diag}(1,-1)$ to solve (4.9). However, for generic systems of Gaudin eigenvalues $E_1, \dots, E_n$, the system of equations (4.9) does not have any solution. The problem to find eigenvalues $E_1, \dots, E_n$ which allow one to construct single-valued solutions of the form (4.8) does not seem to have a direct solution so far. We shall explain in the following that a solution is provided by the uniformization theory of Riemann surfaces with conical singularities.

4.3 Semiclassical behaviour of Liouville correlators

We shall now consider the limit $b \to \infty$ of the Liouville correlators $\Omega^L_{n|n-2}$ which are involved in the correspondence with the H_3^+ model, equation (3.1). The self-duality of Liouville theory implies that this limit is equivalent to the semiclassical limit $b \to 0$. The classical Liouville field φ is now recovered from the quantum Liouville field $\phi = \frac{1}{2}\partial_\alpha V_\alpha\big|_{\alpha=0}$ via $\varphi = \frac{2}{b}\phi$ (instead of $\varphi = 2b\phi$ in the case $b \to 0$). The discussion of the semiclassical limit of Liouville correlators in [15] may be applied to construct the leading asymptotics of Liouville correlation functions in the standard WKB form $e^{-b^2 S^L[\varphi_{\rm cl}]}$, where $S^L[\varphi_{\rm cl}]$ is the Liouville action evaluated on a suitable solution of the classical Liouville equation of motion,

$$\partial_y \partial_{\bar{y}} \varphi_{\rm cl}(y,\bar{y}) = \frac{1}{2} e^{\varphi_{\rm cl}(y,\bar{y})}\,. \tag{4.10}$$

To be more precise, let us note that our Liouville correlator $\Omega^L_{n|n-2}$ contains insertions of $n + (n-2)$ exponential fields V_{α_k} among which the $n-2$ fields $V_{-\frac{1}{2b}}$ are "light" ($\alpha \propto \frac{1}{b}$), whereas the fields V_{α_r}, $r = 1, \dots, n$ are "heavy" in the terminology of [15], since $\alpha_r = b(j_r + 1) + \frac{1}{2b}$ with j_r fixed. The importance of the distinction between light and heavy fields becomes clear when considering the OPE between the Liouville field φ and a generic primary field V_α,

$$\varphi(y,\bar{y}) V_\alpha(z,\bar{z}) \sim -\frac{2\alpha}{b} \log|y-z|^2 + \mathcal{O}(1)\,. \tag{4.11}$$

In the limit $b \to \infty$, the insertion of a light field does not influence the solution of the Liouville equation, whereas insertion of a heavy field implies that the classical solution must diverge near the insertion point z (this divergence makes it necessary to regularize the action S^L, see below). Path integral arguments [15] lead to the following behaviour for $\Omega^L_{n|n-2}$:

$$\Omega^L_{n|n-2} \underset{k\downarrow 2}{\asymp} \exp\big(-b^2 S^L[\varphi_{\rm cl}]\big) \prod_{a=1}^{n-2} e^{-\frac{1}{2}\varphi_{\rm cl}(y_a,\bar{y}_a)}\,, \tag{4.12}$$

where S^L and $\varphi_{\rm cl}$ are defined more precisely as follows:

- The classical solution $\varphi_{\rm cl}(y,\bar{y}|z_1\ldots z_n)$ of the Liouville equation (4.10) is defined by the boundary conditions

$$\varphi_{\rm cl}(y,\bar{y}) \sim -2(j_r+1)\log|y-z_r|^2 + \mathcal{O}(1) \qquad \text{for } y\to z_r\,, \tag{4.13}$$
$$\varphi_{\rm cl}(y,\bar{y}) \sim -2\log|y|^2 + \mathcal{O}(1) \qquad \text{for } y\to\infty\,. \tag{4.14}$$

  Existence and uniqueness of a solution to the Liouville equation is guaranteed by the uniformization theorem [23, 24] for the Riemann surface

$$\Sigma_n \equiv \mathbb{P}^1 \setminus \{z_1,\ldots,z_n\}\,,$$

  with conical singularities of order j_r+1 at the points $z_1,\ldots,z_n$.

- The Liouville action $S^L[\varphi_{\rm cl}]$ is regularized as follows: $S^L[\varphi] \equiv \lim_{\epsilon\to 0} S^L_\epsilon[\varphi]$, where

$$\begin{aligned} S^L_\epsilon[\varphi] = {} & \frac{1}{4\pi}\int_{X^\epsilon} d^2z\Big(|\partial_z\varphi|^2 + e^\varphi\Big) + \left(\frac{\epsilon}{2\pi}\int_{\partial D_{n+1}} dx\,\varphi - 2\log\epsilon\right) - \\ & -\sum_{r=1}^{n}\left(\frac{j_r+1}{2\pi\epsilon}\int_{\partial D_r} dx\,\varphi + 2(j_r+1)^2\log\epsilon\right)\,, \end{aligned} \tag{4.15}$$

  where $D_r = \{z\in\mathbb{C}; |z-z_r|<\epsilon\}$, $D_{n+1} = \{z\in\mathbb{C}; |z|>1/\epsilon\}$, and $X^\epsilon = D_{n+1} \setminus \bigcup_{r=1}^n D_r$.

4.4 Gaudin eigenvalues from accessory parameters

Let us now deduce the asymptotic behaviour of the H_3^+ correlator Ω^H_n from the Liouville theory result (4.12) through the H_3^+-Liouville correspondence.

First, the behaviour is indeed of the form (4.7), with only a single potential $S(z_1\ldots z_n)$. The potential which appears is

$$S = S^L[\varphi_{\rm cl}]\,. \tag{4.16}$$

Moreover, the corresponding eigenvector is

$$\Psi = \delta^{(2)}\left(\sum_{r=1}^{n}\mu_r\right)|u|^2 \prod_{a=1}^{n-2} e^{-\frac{1}{2}\varphi_{\rm cl}(y_a,\bar{y}_a)}\,. \tag{4.17}$$

Let us now find out which objects from Liouville theory correspond to the eigenvalues $E_1\ldots E_n$. First we rewrite the Liouville equation in a Fuchsian form, similar to the Gaudin eigenvalue equations with separated variables (4.5). For this we use the classical energy-momentum tensor T_φ defined as

$$(\partial_y^2 - T_\varphi)e^{-\frac{1}{2}\varphi} = 0 \quad \Leftrightarrow \quad T_\varphi = -\frac{1}{2}(\partial_y\varphi)^2 + \partial_y^2\varphi\,. \tag{4.18}$$

The quantity T_φ thus defined is holomorphic iff φ satisfies the Liouville equation (4.10). If φ is furthermore taken to be the unique classical solution with singular behavior specified in (4.13) then its behaviour near the singularities (including $y = \infty$) implies that T_φ can be expressed as

$$T_\varphi(y|z_1 \dots z_n) = \sum_{r=1}^{n} \left(\frac{-j_r(j_r+1)}{(y-z_r)^2} + \frac{C_r(z_1 \dots z_n)}{y - z_r} \right). \tag{4.19}$$

The functions C_r have become famous under the name of accessory parameters. The regularity of φ near $y = \infty$ requires the three conditions

$$\sum_{r=1}^{n} z_r^k \big(z_r C_r - (k+1) j_r(j_r+1) \big) = 0 \qquad \text{for } k = -1, 0, 1\,. \tag{4.20}$$

Otherwise it is difficult to determine the accessory parameters more explicitly. Nevertheless, they can be shown [21, 22] to be related to the classical Liouville action of the solution $\varphi_{\rm cl}$ via

$$C_r = -\partial_{z_r} S^L\,, \qquad \bar{C}_r = -\partial_{\bar{z}_r} S^L\,. \tag{4.21}$$

These observations immediately imply that the function Ψ from equation (4.17) is an eigenvector for the Gaudin hamiltonians in separated variables, with eigenvalues

$$E_r = -C_r\,, \qquad \bar{E}_r = -\bar{C}_r\,, \qquad r = 1 \dots n\,. \tag{4.22}$$

It seems quite remarkable that the uniqueness of the solution $\varphi_{\rm cl}$ to the uniformization problem implies the uniqueness of the choice (4.22) for the eigenvalues $E_1, \dots, E_r$, thus uniquely solving the "Gaudin single-valued eigenvector problem" that was formulated in subsection 4.2. This relation is worth further investigation. In particular, we would like to understand its representation-theoretic origins in more detail. This seems to require some generalization of the discussion in [20].

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A. The H_3^+ model with the standard x variables

In our review of the H_3^+ model in the main text, the standard basis $\Phi^j(x)$ of fields has been replaced with the new basis,

$$\Phi^j(\mu|z) = \frac{1}{\pi}|\mu|^{2j+2}\int_{\mathbb{C}} d^2x\ e^{\mu x-\bar{\mu}\bar{x}}\Phi^j(x|z)\,, \tag{A.1}$$

which is more convenient for our correspondence with Liouville theory. In this appendix we explain how to derive the structure constants of the model with μ variables from the well-known expressions with x variables.

The OPE of the H_3^+ model in the x variables reads

$$\begin{aligned}\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) = &\int_{-\frac{1}{2}+i\mathbb{R}_+} dj_3 \int d^2x_3 \big\langle \Phi^{j_3}(x_3|\infty)\Phi^{j_2}(x_2|z_2-z_1)\Phi^{j_1}(x_1|0)\big\rangle \times \\ &\times\big(\Phi^{-j_3-1}(x_3|z_1)+(\text{descendants})\big)\,. \end{aligned}\tag{A.2}$$

The construction of n-point functions is thereby reduced to the construction of two- and three-point functions:

$$\begin{aligned}\langle\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1)\rangle &= |z_2-z_1|^{-4\Delta_j}\big(\delta^{(2)}(x_2-x_1)\delta(j_2+j_1+1)+ \\ &\quad +B^H(j_1)|x_2-x_1|^{4j_1}\delta(j_2-j_1)\big) \end{aligned}\tag{A.3}$$

$$\begin{aligned}\langle\Phi^{j_3}(x_3|z_3)\cdots\Phi^{j_1}(x_1|z_1)\rangle &= |z_3-z_2|^{2\Delta_{32}^1}|z_3-z_1|^{2\Delta_{31}^2}|z_2-z_1|^{2\Delta_{21}^3}\times \\ &\quad\times|x_3-x_2|^{2j_{32}^1}|x_3-x_1|^{2j_{31}^2}|x_2-x_1|^{2j_{21}^3}\times \\ &\quad\times C^H(j_3,j_2,j_1)\,, \end{aligned}\tag{A.4}$$

where the structure constant $C^H(j_3,j_2,j_1)$ has been given in equation (2.28), and the constant $B^H(j)$ is

$$B^H(j) = \frac{1}{\pi b^2}\frac{(-\pi b^2\gamma(-b^2))^{2j+1}}{\gamma(-b^2(2j+1))}\,. \tag{A.5}$$

The reflection coefficient $R^H(j)$ (2.31) can be related to the constant $B^H(j)$ by applying the Fourier transform to the two-point function of the model. The relation is:

$$R^H(j) = B^H(j)\times|\mu|^{4j+2}\int_{\mathbb{C}} d^2x\, e^{\mu x-\bar{\mu}\bar{x}}|x|^{4j} = B^H(j)\times\pi\gamma(2j+1)\,. \tag{A.6}$$

Note that the integral over x diverges. Thus this integral has to be regularized or interpreted in terms of distributions.

The structure constant D^H (2.29) can similarly be related to the x-dependent factors in the three-point function eq. (A.4):

$$\begin{aligned}\delta^{(2)}(\mu_1+\mu_2+\mu_3)D^H\left[\begin{smallmatrix} j_3 & j_2 & j_1\\ \mu_3 & \mu_2 & \mu_1\end{smallmatrix}\right] &= \frac{1}{\pi^3}\prod_{i=1}^3|\mu_i|^{2j_i+2}\int_{\mathbb{C}} d^2x_3d^2x_2d^2x_1\times \\ &\quad\times e^{\sum_r(\mu_r x_r-\bar{\mu}_r\bar{x}_r)}|x_{12}|^{2j_{12}^3}|x_{13}|^{2j_{13}^2}|x_{23}|^{2j_{23}^1}\,. \end{aligned}\tag{A.7}$$

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The computation can be performed as follows: first change $x_{1,2} \to x_{1,2} + x_3$ so that the integral over x_3 can be performed and yields a $\pi^2\,\delta^{(2)}(\mu_1+\mu_2+\mu_3)$ prefactor. Then, perform the changes of variables $x_2 = \lambda x_1$ and then $x_1 \to x_1/(\mu_1+\lambda\mu_2)$ to obtain

$$\begin{aligned} D^H &= \frac{1}{\pi}\prod_{i=1}^{3}|\mu_i|^{2j_i+2}\int d^2x_1\, e^{\mu_1 x_1-\bar\mu_1\bar x_1}|x_1|^{2(\sum j+1)}\times \\ &\quad\times\int d^2\lambda|\mu_1+\lambda\mu_2|^{-2(\sum j+2)}|\lambda|^{2j^1_{23}}|1-\lambda|^{2j^3_{12}}\,. \end{aligned} \tag{A.8}$$

The first integral has already been used in equation (A.6) above. The integral over λ was computed by Dotsenko [26] in the following form:

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{C}} d^2x|x|^{2a}|1-x|^{2b}|x-y|^{2c} &= \frac{\gamma(b+1)\gamma(-c-a-b-1)}{\gamma(-a-c)}\times \\ &\quad\times{}_2\mathcal{F}_1(-c,-c-1-a-b,-c-a,y)+ \\ &\quad+\frac{\gamma(a+1)\gamma(c+1)}{\gamma(a+c+2)}|y|^{2(a+c+1)}{}_2\mathcal{F}_1(-b,a+1,a+c+2,y)\,, \end{aligned} \tag{A.9}$$

where the notation ${}_2\mathcal{F}_1$ was defined in equation (2.30).

B. Special functions

The function $\gamma(x)$ is built from Euler's Gamma function:

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{B.1}$$

The function Υ_b is defined for $0 < \Re x < Q$ by

$$\log\Upsilon_b = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \tag{B.2}$$

This function can be extended to a holomorphic function on the complex plane thanks to the shift equations

$$\Upsilon_b(x+b) = \gamma(bx)b^{1-2bx}\Upsilon_b(x)\,, \qquad \Upsilon_b\left(x+\frac{1}{b}\right) = \gamma\left(\frac{x}{b}\right)b^{2x/b-1}\Upsilon_b(x)\,. \tag{B.3}$$

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